Lecture 24

Wednesday, 23 November 2022 11:41 AM

STABLE MATCHINGS

- set M of men, W of women, $|M| = |W| = n$
- each $m \in M$ has a strict preference order over W
- each $w \in W$ has " " " " " M

$w_1 >_m w_2 \Rightarrow$ man m prefers w_1 to w_2

$m_1 >_w m_2 \Rightarrow$ woman w prefers man m_1 to m_2

- a matching $\pi : M \to W$ is a bijection (will abuse notation & use both $\pi(m)$ & $\pi(w)$ to denote partners in the matching)

Defn (Blocking Pair): Given a matching π, a pair (m, w) is a blocking pair if:

$w >_m \pi(m)$ AND $m >_w \pi(w)$

(i.e., m & w are better off together, than with their partners in π)

Defn (Blocking Coalition): Given a matching π, a subset of agents $S \subseteq M \cup W$ is a blocking coalition if $\exists$ a matching π' s.t.

$\forall m \in S$, $\pi'(m) \in S$ & $\forall w \in S$, $\pi'(w) \in S$

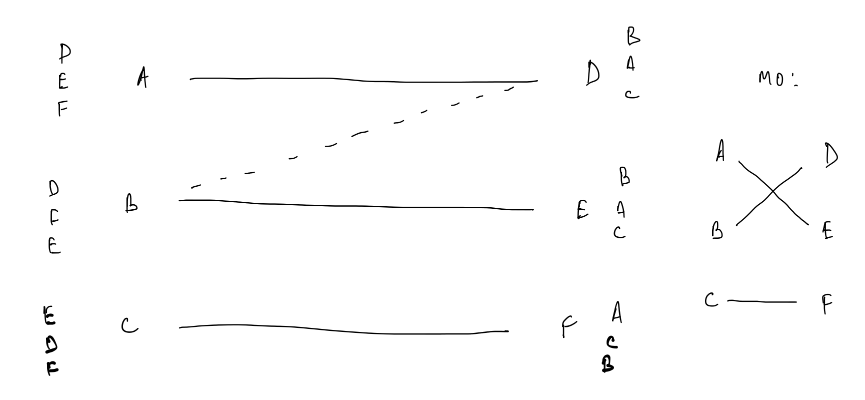
$\forall w, m \in S$, $\pi'(m) >_m \pi(m)$ & $\pi'(w) >_w \pi(w)$

But let's focus on avoiding blocking pairs for now.

Applications: (i) School applications
(ii) Assigning interns to hospitals
(iii) Engineering entrance in India

Example:

M:



(B,D) are a blocking pair in the current matching.

Q. Consider the simple algorithm that starts w/ an arbitrary matching π. If (m,w) is a blocking pair, then it removes the edges $(m, \pi(m))$ & $(w, \pi(w))$, & instead adds the edges (m, w) & $(\pi(m), \pi(w))$. Does this algo produce a stable matching?

Gale - Shapley Stable Matching Algo (Man - Proposing):

- $L(m) = W \quad \forall m \in M$
- $\tau(m) = \tau(w) = \phi \quad \forall m, w$ (τ is a partial, temporary matching)
- while $\exists m : L(m) \neq \phi$ and $\tau(m) = \phi$
  - let $w \in L(m)$ be m's most preferred agent
  - $L(m) \leftarrow L(m) \setminus w$ // m "proposes" to w
  - if $\tau(w) = \phi$
    $\tau(w) \leftarrow m$, $\tau(m) \leftarrow w$ // w "temporarily accepts m's proposal"
  - else if $\tau(w) = m'$ and $m >_w m'$
    $\tau(w) \leftarrow m$, $\tau(m) \leftarrow w$, $\tau(m') \leftarrow \phi$ // Same as above, & w rejects m'
  - else if $\tau(w) = m'$, and $m' >_w m$
    continue // w rejects m
- return $\pi = \tau$

We say a "round" is an execution of the while loop.

Note: (i) τ is a partial matching (i.e. if $\tau(m) = w$, $\tau(w) = m$)
(ii) once w accepts a proposal, $\tau(w) \neq \phi$ subsequently
(iii) if $\tau(w) = m$ in some round, then in any later round, $\tau(w) \geq_w m$

Theorem: Algo terminates in $\leq n^2$ rounds w/ a stable matching

Proof: (i) terminates in n^2 rounds:
- in each round, $L(m)$ decreases by 1 for some $m \in M$
- hence after n^2 rounds, $L(m) = \phi$ for all $m \in M$.

(ii) with a matching

Say $\exists m : \pi(m) = \phi$. Since algo terminated, $L(m) = \phi$, hence m was rejected by all the women.

Suppose r_i is the round when m was rejected by w_i, and say $r_1 > r_2 > \ldots > r_n$

But, then w_1 was matched in all rounds after r_1

w_2 " " " " " " r_2

...

So after r_n, all women were matched

But since $\tau(m) = \phi$, there are only $n-1$ men they could be matched to.

(iii) with a stable matching.

For a contradiction, say (m,w) is a blocking pair.

Since $w >_m \pi(m)$, m must have proposed to w before $\pi(m)$.

Since he got rejected, $\tau(w) >_w m$ in that round.

Hence, $\pi(w) \geq_w \tau(w) >_w m$, and (m,w) cannot be a blocking pair. ∎

Note that the matching obtained by the algo is not the only stable matching possible; there could be many others.

Eg., we could reverse the roles of men & women, and have women proposing, which would possibly give a different stable matching.

However the matching obtained by the algo is special: it is the unique man - optimal matching, in the following sense:

Consider all possible stable matchings.

For each $m \in M$, let $h(m)$ be the most-preferred woman m is matched to, in all the stable matchings.

Theorem: $\pi(m) = h(m) \quad \forall m \in M$

Proof: Fix a run of the algo.

Let $R = \{(m,w)\}$ be all "rejected" pairs in this run.

Claim: If $(m,w) \in R$ then (m,w) are not matched in any stable matching.

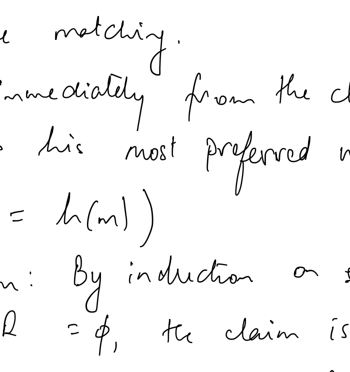
(it follows immediately from the claim that m must be matched to his most preferred woman in any stable matching, i.e., $\pi(m) = h(m)$)

Proof of claim: By induction on size of R.

If $R = \phi$, the claim is true.

Suppose the claim holds for the current set R, & in some round, w rejects m (so (m,w) is added to R_m).

BUT $\exists$ stable matching $\sigma : \sigma(m) = w$



Since w rejects m, $\tau(w) >_w m'$

Since σ is stable, (m', w) is not a blocking pair, hence if $w' = \sigma(m')$, then $w' = \sigma(m') >_{m'} w$

So in algo, m' must have proposed to w', & got rejected, before he got attached to w.

But then (m', w') must have been added to R earlier, & $\sigma(m') = w'$, a contradiction to the I.H. ∎